CONTINUITY OF FUNCTIONS

DEFINITION OF CONTINUITY AT A POINT

Let $f: D \longrightarrow \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}$, and let $x_0 \in D$.

- If x_0 is an accumulation point of D, then f is said to be **continuous at x_0** if $\lim_{x\to x_0} f(x) = f(x_0)$.

  This means that:

  1. both the left-hand limit and the right-hand limit of f at x_0 exist and are finite:

     $$\lim_{x\to x_0^-} f(x) = l_1 \in \mathbb{R} \qquad \wedge \qquad \lim_{x\to x_0^+} f(x) = l_2 \in \mathbb{R}$$

  2. $l_1 = l_2 = \lim_{x\to x_0} f(x) = l$
  3. $f(x_0) = l$.

- If x_0 is an isolated point of D, then f is said to be continuous at x_0 by definition.

DISCONTINUITY POINTS

Let $f: D \longrightarrow \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}$, and let x_0 be an accumulation point of D.

x_0 is said to be a **discontinuity (point)** of f if either:

- $x_0 \in D$, but f is not continuous at x_0 , or
- $x_0 \notin D$ (i.e. f is not defined at x_0).

CLASSIFICATION OF DISCONTINUITY POINTS

I. x_0 is called a **discontinuity of the first kind**, or a **jump discontinuity**, or a **step discontinuity**, if:

both the left-hand limit and the right-hand limits of f at x_0 exist and are finite:

$$\lim_{x\to x_0^-} f(x) = l_1 \in \mathbb{R} \qquad \wedge \qquad \lim_{x\to x_0^+} f(x) = l_2 \in \mathbb{R},$$

but $l_1 \neq l_2$.

The number $s = |l_2 - l_1|$ is called the **jump size** of f at x_0.

<u>Example</u>

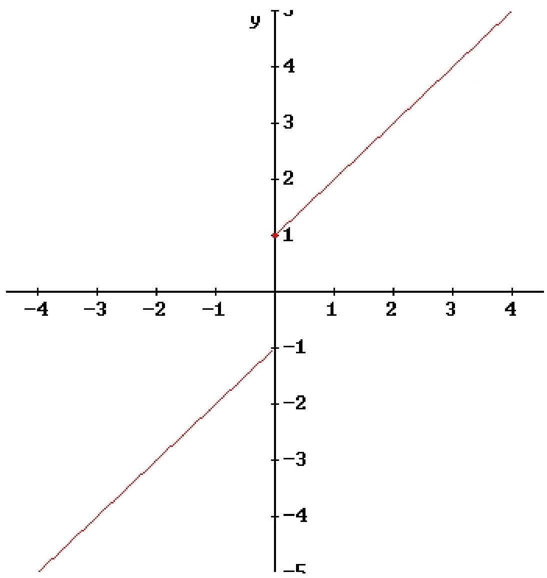
Consider the function $f(x) = \begin{cases} x+1, & se\ x \geq 0 \\ x-1, & se\ x < 0 \end{cases}$.

It is defined $\forall x \in \mathbf{R}$.

At $x_0 = 0$, in which it assumes the value $f(0) = 1$, we have:

$\lim_{x\to 0^-} f(x) = -1$ and $\lim_{x\to 0^+} f(x) = 1$.

Therefore $x_0 = 0$ is a discontinuity point of the first kind, with jump size $s = 2$.



II. x_0 is called a **discontinuity of the second kind,** or an **essential discontinuity**, if:

either the left-hand limit or the right-hand limit of f at x_0 does not exist or is infinite.

<u>Example 1</u>

Consider the function $f(x) = \frac{x}{x+1}$.

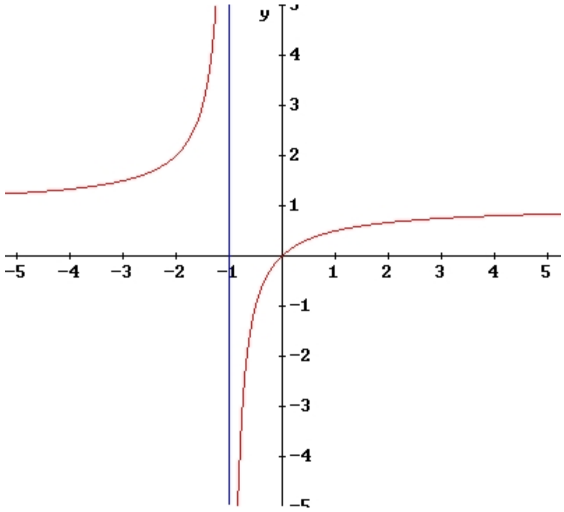
It is defined $\forall x \neq -1$, so it has a discontinuity at $x_0 = -1$.

Since

$\lim_{x\to -1^-} f(x) = +\infty$ and $\lim_{x\to -1^+} f(x) = -\infty$,

$x_0 = -1$ is a discontinuity point of the second kind.

The line $x = -1$ is a vertical asymptote of the graph of f as well.



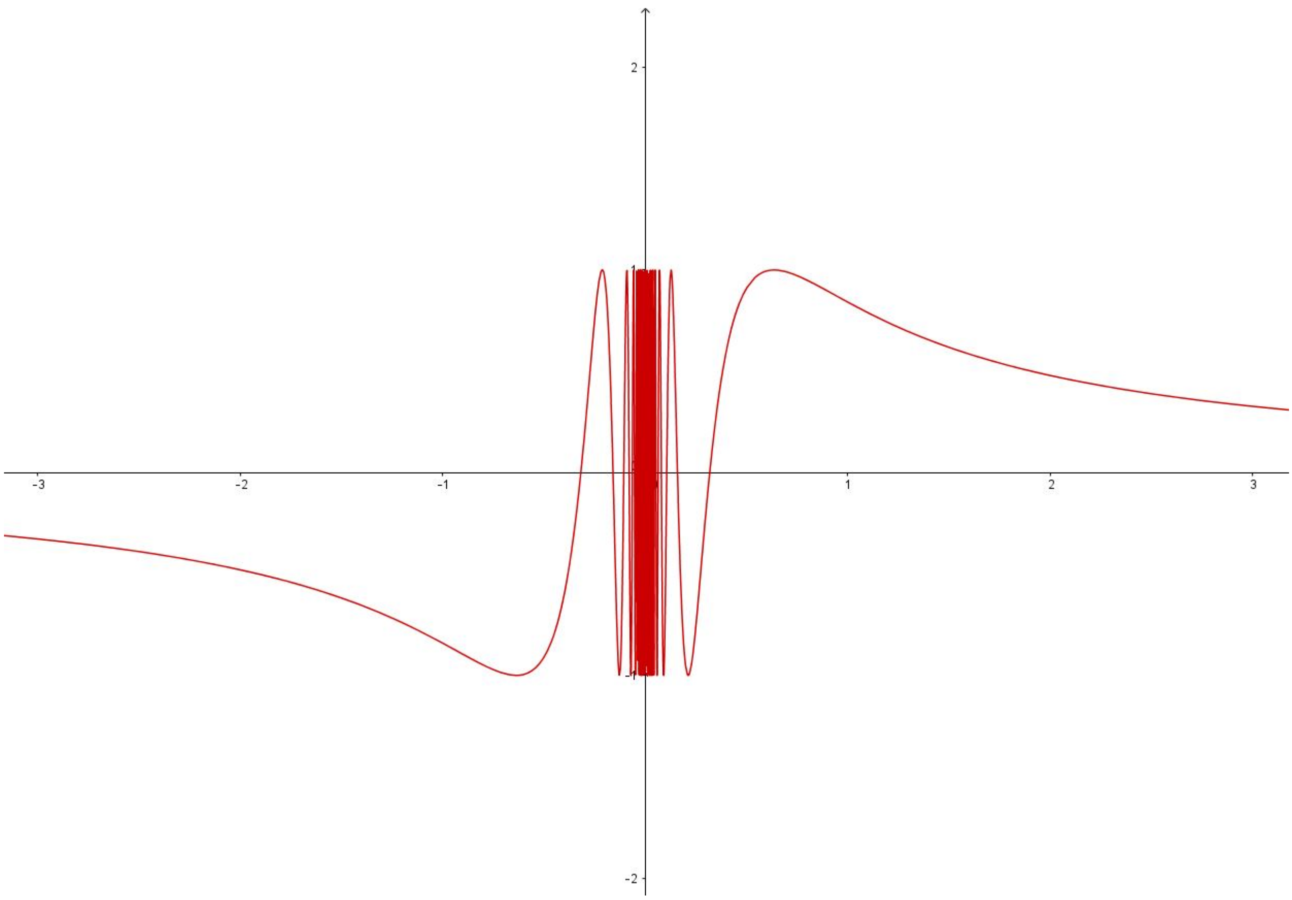
<u>Example 2</u>

Consider the function $f(x) = \sin\frac{1}{x}$.

It is defined $\forall x \neq 0$, so it has a discontinuity at $x_0 = 0$.

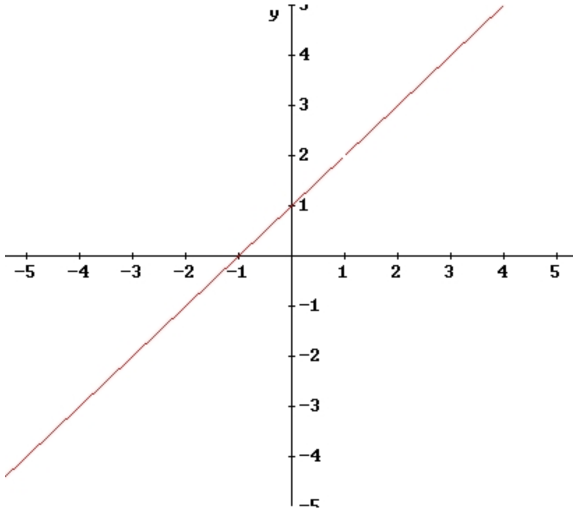
Because $\lim_{x\to 0} f(x)$ does not exist,

$x_0 = 0$ is a discontinuity point of the second kind.



III. x_0 is called a **discontinuity of the third kind,** or a **removable discontinuity**, if:

there exists $\lim_{x\to x_0} f(x) = l \in \mathbb{R}$, but either f is not defined at x_0 or $f(x_0) \neq l$.



Example

Consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$.

It is defined $\forall x \neq 1$, so it has a discontinuity at $x_0 = 1$. Observe that, $\forall x \neq 1$,

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{(x-1)} = x + 1.$$

Therefore

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 2$$

and $x_0 = 1$ is a discontinuity point of the third kind.

Note that in the graph of f there is a "hole" at $x_0 = 1$, so it's a line from which a point has been removed.