THE RIGHT TRIANGLE THEOREMS

FIRST THEOREM

In a right triangle, the length of a side is equal to the length of the hypotenuse, multiplied by:

- the sine of the angle opposite the side

  or

- the cosine of the angle adjacent to the side.

SECOND THEOREM

In a right triangle, the length of a side is equal to the length of the other side, multiplied by:

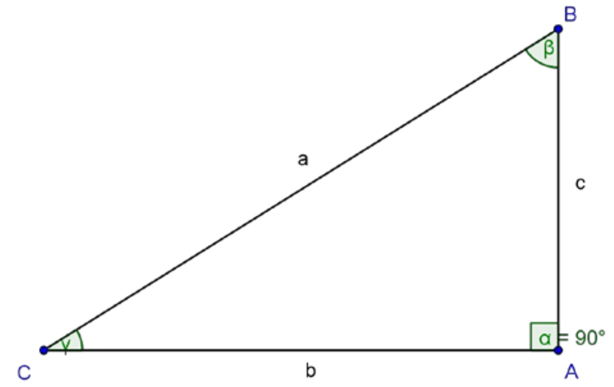
- the tangent of the angle opposite the first side

  or

- the cotangent of the angle adjacent to the first side.

<table>
<tr><td>1st THEOREM

$b = a\sin\beta = a\cos\gamma$

$c = a\sin\gamma = a\cos\beta$</td><td rowspan="2">
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<tr><td>2nd THEOREM

$b = c\tan\beta = c\cot\gamma$

$c = b\tan\gamma = b\cot\beta$</td></tr>
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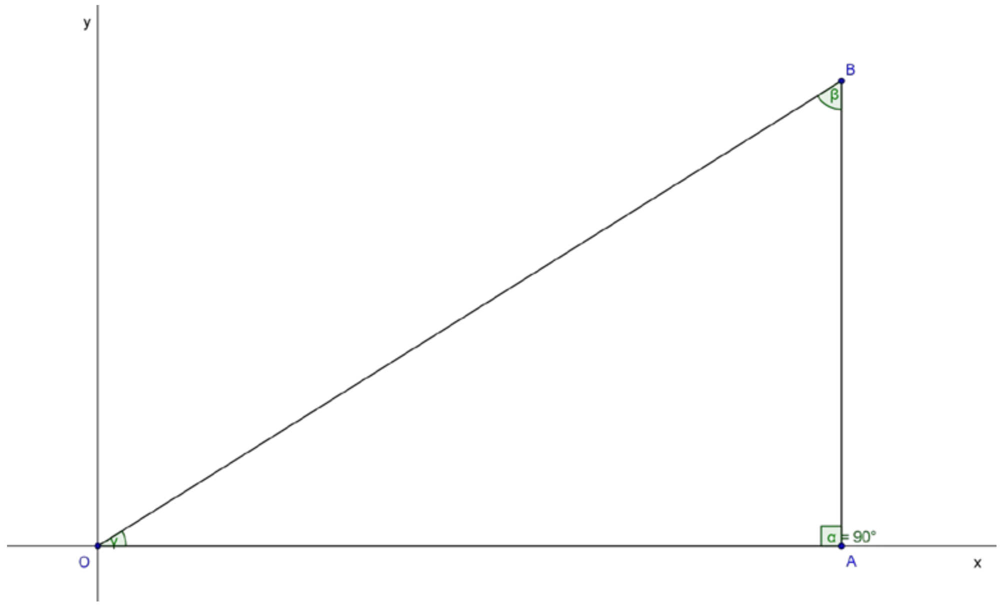
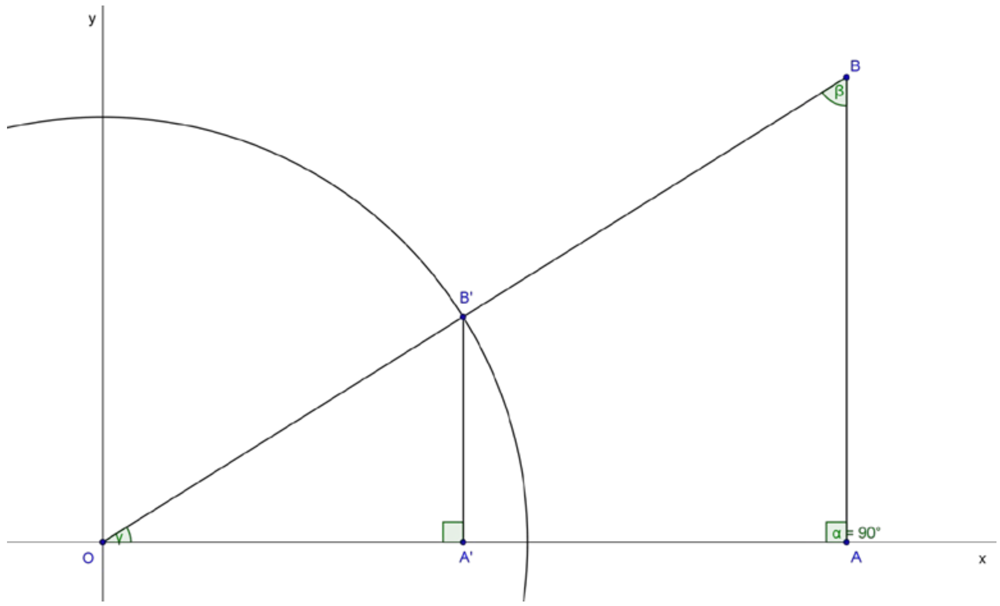
Proof.

To prove the theorems, we place a right triangle in a coordinate plane, with one side lying on the positive x-axis and its adjacent acute angle in standard position. So one acute-angle vertex is at the origin O. Let's call the the right-angle vertex A and the other acute-angle vertex B.



Now, let's draw the unit circle. Clearly, we don't know in general whether each side of the triangle is greater or less than the unit radius (it depends on their lengths), but it doesn't matter, because the theorem we are going to prove holds, and its proof works, in any case. So let's suppose, for instance, that $OA > 1$. Let B' be the intersection point between the hypotenuse of the triangle and the unit circle and let A' be its projection on the x-axis.



Now consider the two triangles OAB and $OA'B'$.

They are similar because they are both right triangles and they share the angle $\gamma = A\hat{O}B = A'\hat{O}B'$, so the third angle of the first triangle is congruent to the corresponding angle of the second triangle, too.

As a result, the corresponding sides of the two triangles are in the same ratio. Specifically:

$$\frac{AB}{A'B'} = \frac{OA}{OA'} = \frac{OB}{OB'}$$

(read AB **is to** $A'B'$ **as** OA **is to** OA' **as** OB **is to** OB').

Now recall that, by definition, $\overline{OB'} = 1$, $\overline{OA'} = \cos\gamma$ and $\overline{A'B'} = \sin\gamma$, hence the previous proportion becomes:

$$\frac{AB}{\sin\gamma} = \frac{OA}{\cos\gamma} = \frac{OB}{1}.$$

Consider the first and third ratios; solving for AB gives:

$$\overline{AB} = \overline{OB}\sin\gamma.$$

Consider the second and third ratios; solving for OA gives:

$$\overline{OA} = \overline{OB}\cos\gamma.$$

These equalities prove the first theorem.

Consider the first two ratios; solving for AB gives:

$$\overline{AB} = \overline{OA}\frac{\sin\gamma}{\cos\gamma} = \overline{OA}\tan\gamma\,;$$

solving for OA gives:

$$\overline{OA} = \overline{AB}\frac{\cos\gamma}{\sin\gamma} = \overline{AB}\cot\gamma\,;$$

which prove the second theorem.